Lecture 13: Solving systems of linear equations (Gaussian elimination)

November 16, 2016 8:48 PM

<u>11.1</u>
Sometimes it's hard to determine linear dependency if the numbers aren't elementary. Learning the following technique for solving linear equations will bypass this issue.

<u>Example</u>
Is $\{(1,2,3), (4,5,6), (7,8,9)\}$ linearly dependent or independent?

We want to look at solutions to the following equation:
$a(1,2,3) + b(4,5,6) + c(7,8,9) = (0,0,0)$

We can rewrite this:
$a + 4b + 7c = 0$
$2a + 5b + 8c = 0$
$3a + 6b + 9c = 0$

coefficients

homogeneous linear system:
when all RHS (right hand sides) are 0

This is a linear system with $m = 3$ equations and $n = 3$ variables.
When we evaluate a linear system, we're looking to find the set S of all solutions (a, b, c) that satisfy all the equations above *simultaneously*. The set S is a **general solution** to the linear system.

<u>11.2 Examples and vocabulary</u>

a)
$a + 2b = 7$ ①
$2a - b = 4$ ②

$a + 2b = 7$
②-2① → $-5b = -10$

$b = 2$
$a = 3$
$\Rightarrow S = \{(3,2)\}$

"general solution"

b)
$a + 2b = 7$
$-a - 2b = 1$

$a + 2b = 7$ contradiction
$0 = 8$ ↯

$\Rightarrow S = \{\} = \emptyset$

c)
$a + 2b = 7$
$3a + 6b = 21$

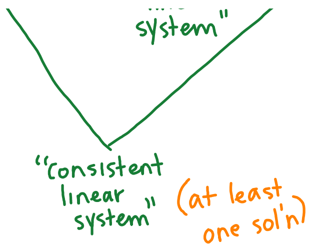
$a + 2b = 7$
$0 = 0$
==========
Assign a free parameter
$b = t$
$a = 7 - 2t$
$\Rightarrow S = \{(7 - 2t, t) | t \in \mathbb{R}\}$
$\Rightarrow S = \{(7,0) + t(-2,1) | t \in \mathbb{R}\}$

degenerate equations

inhomogeneous solution:
not all RHSs are 0

one solution **no solution** **infinitely many solutions**

"inconsistent linear system"



11.3 Observations

a) Homogenous linear systems are **always** consistent because $(0, \dots, 0)$ is always a possible solution.
b) Any linear system with a degenerate inhomogeneous equation is inconsistent.

Theorem

Any ***linear*** system has either 0, 1, or infinitely many solutions.

11.4 Elementary row operators

In examples a), b), and c) above, we added multiples of one row to another without changing the general solution. This allows us to simplify a system so that we can obtain a general solution.
The following **elementary row operators** (ERo)are possible and will not alter the general solution:

(1) Add/subtract a multiple of one row to another
(2) Interchange two rows
(3) Multiply a row by a non-zero scalar.

note:

- any solution before an Ero will still be a solution after an Ero
- every ERo is reversible

Therefore, EROs will not change the general solution.
(see attached handout for Gaussian Elimination algorithm to follow)

Example

"augmented matrix"

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$$

coefficient matrix

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \quad -\tfrac{1}{3}R_2 \to R_2 \qquad \text{“}\sim\text{”} = \text{row equivalent}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \quad R_3 + 6R_2 \to R_3$$

leading ones

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

t (free parameter)

$b = -2t$
$a = -4b - 7 = 8t - 7t$

So,

$$S = \left\{ \begin{pmatrix} t \\ -2 \\ t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

11.5 Row Echelon Form (**REF**) and Reduced Row Echelon Form

(RREF)

Definition:

A matrix (augmented or not) is in **row echelon form** (**REF**) if:

i) all zero rows are at the bottom
ii) the first non-zero entry in each row is a 1
iii) each leading 1 is to the right of the leading 1's in the row above

A matrix is in **reduced row echelon form** (**RREF**) if, in addition:

iv) each leading 1 is the only non-zero entry in its column

Reduced row echelon form (RREF) is better because we can read off a general solution immediately.

Example

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 - 4R_2 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \boxed{-1} & 0 \\ 0 & 1 & \boxed{2} & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right] \quad \text{RREF}$$

$$\begin{array}{ccc} a & b & c=t \end{array}$$

column w/o leading 1 is a free parameter

Therefore:

$\Rightarrow a - t = 0 \quad \Rightarrow \quad a = t$

$\quad b + 2t = 0 \qquad\; b = -2t$

$$\Rightarrow S = \left\{ \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

Let's write down the process for Gaussian elimination. It can be applied to any matrix C, and stops at a matrix $\tilde{C}$ which is in RREF.

Step 1 If the matrix C is zero, stop.

Step 2 Locate the left-most nonzero column, and interchange the top row with another, if necessary, to bring a non-zero entry to the first row of this column.

Step 3 Scale the first row, as necessary, to get a leading 1.

Step 4 If necessary, annihilate the rest of the column BELOW using this leading 1 as a pivot. That is, if a_i is the entry in this column of Row i, then add $-a_i R_1$ to R_i, and put the result back in the i^{th} row.

Step 5 This completes the operations (for now) with the first row. If there was only one row in your matrix, at this stage, stop. Otherwise, *ignore the first row* (but don't lose it!) and go back to step 1.

When this stops, the matrix you have will be in REF. Now proceed with the following steps to put the matrix in RREF:

Step 6 If the *right most* leading 1 is in row 1, stop.

Step 7 Start with the *right most* leading 1 – this will be in the last non-zero row. Use it to annihilate every entry *above* it in its column. That is, if $a_i \neq 0$ is the entry in this column in row i, then add $-a_i$ times this row to R_i, and put the result back in the i^{th} row.

Step 8 Cover up the row you used and go to step 6.